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Investigation of Efficiency of Stochastic Differential Equations Driven by Levy Process in Modeling of Exchange Rate Volatility (COGARCH Approach)

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Abstract
Exchange rate and its volatility plays a significant role in financial decisions and economic transactions affected by large and small economic groups. Present study aims to provide continuous modeling for exchange rate data in Iran with the support of a stochastic differential equation driven by the Levy process (that named continuous GARCH model) and check out the fitting exchange rate volatility on this model. Accordingly, we use the daily data of the unofficial exchange rate (the value of the US dollar against the Iranian Rial in the free market) from March 2009 to March 2018. We also challenge the performance of the models with time-varying volatility under the continuous features in comparison to the discrete GARCH model. Finally, according to investigating the efficiency of this model coinciding with the results of the measurement error criterion, the preference of the new continuous model is expressed.

Keywords: Stochastic Differential Equations, Levy Process, GARCH Model, Continuous GARCH Model, Exchange Market

JEL Classification: C22 ، C29 ، C58 ، C59 ، E44.

1. Introduction
Simultaneously with the expansion of the financial markets, especially the exchange market, developing countries have often suffered from the turmoil of the foreign exchange market and it is important to choose policies for macroeconomic stability. Therefore, the exchange market, like any other market, can evade economic momentum and undergo changes.

The necessity of recognizing the volatility of exchange rates is important because the prediction and orientation of its volatility are a key factor in the pricing of derivatives, the calculation of value at risk and determining the optimal cost ratios. Moreover, it is very important to foresee it for financial institutions to evaluate the risk of exchange rate, increase the profits and monitor financial planning. Also, governments, economists and consumers in the exchange market are increasingly interested in models that enable accurate prediction of the exchange rates (Lahmiri, 2017).

The present study is an attempt to model the exchange rate volatility on the basis of a stochastic differential equation under the Levy process. In this

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modeling, a continuous model based on a discrete version of the GARCH process is used, and its parameters are estimated by the quasi maximum likelihood estimation. According to the results of the research, the volatility in the Iranian exchange market follows the continuous model and this model has been successful in capturing volatility.

2. Background

There are a variety of models for achieving the purpose of this research in the field of financial mathematics and stochastic processes. Stochastic differential equations are widely used in complex modeling of probability, such as market volatility modeling. The modeling of the economic and financial variables was initiated by Osborne (1964) and Samuelson (1965). It was first introduced into the economic literature by the Black-Scholes option pricing theory (1973) and simultaneously with Merton (1973) for modeling stock price in the form of a stochastic differential equation.

Besides modeling with random differential equations, well-known time series models such as ARCH and GARCH are appropriate models for modeling time series data. These models are widely used for modeling volatility and demonstratng the stability of volatility in the very high levels. The ARCH models were introduced by Engle (1982) and were then extended by Bollerslev (1986) to GARCH. Nelson (1986), finally, extended it to EGARCH (exponential GARCH) (Nelson, 1990–b).

To model the stochastic volatility, there are different approaches to finding a continuous model. The first time Nelson (1990) attempted to extend the modeling of time series data from discrete to the continuous form (Nelson, 1990-a). After Nelson, Barndorff-Nielsen and Shepard (2001) developed a stochastic volatility model whose variation was driven under the continuous Levy process $(L_t)_{t\geq 0}$. Then, Kluppelberg et al. (2004) presented the idea of a new continuous model derived from a discrete GARCH model, where the price equation and its volatility process are a stochastic differential equation under the continuous Levy process. Their new and practical model in financial literature has been developed for continuous GARCH (COGARCH) model and has been the basis for this research.

3. Method

In this study, following the method used by Kluppelberg et al., we first estimated the parameters of the GARCH (1,1) discrete time process parameters developed by Bollerslev (1986):

$$y_n = \varepsilon_n \sigma_n \tag{1}$$

$$\sigma_n^2 = \beta + \lambda y_{n-1}^2 + \delta \sigma_{n-1}^2 \quad , \quad n \epsilon N \tag{2}$$

Then, using the Levy process, we defined the continuous time GARCH process as the following stochastic differential equation:

$$dG_t = \sigma_{t-} dL_t \ , \ \ G_0 = 0 \tag{3}$$

$$d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)dt + \varphi\sigma_{t-}^2 d[L, L]_t^d \ . \tag{4}$$

After estimating the parameters of the new continuous model and modeling, to achieve the next goal of the paper, the volatility recursive estimator was calculated to compare the volatility of the two models:

$$\hat{\sigma}_t^2 = \hat{\beta} + (1 - \hat{\eta})\hat{\sigma}_{t-1}^2 + \hat{\varphi}\left(G_t^{(1)}\right)^2 \qquad n\epsilon N \tag{5}$$

That is, $\left(G_t^{(1)}\right)^2$ is the return on the exchange rate . $\hat{\sigma}_0^2$ was also obtained from equation $\hat{\sigma}_0^2 = \frac{\hat{\beta}}{\hat{\eta} - \hat{\varphi}}$.

Then, for comparison purposes, we used Root mean squared error (RMSE) and Mean absolute error (MAE) and Mean heteroscedastic squared error (MHSE) for model estimation performance.

4. Estimation and analysis

In the present study, 2539 daily data were collected from March 2009 to March 2018 in Iran, which were obtained from central bank data and official exchange bureaus. The figures drawn from these time series showed an increase in the price of exchange rate in the 8-year period, as well as a sudden rise in prices in the middle of the period. Due to the negative skewness and positive kurtosis of exchange rate data, the non-normality of exchange rate data was determined during this period of eight years, which can be evidence of the presence of volatility in the data set.

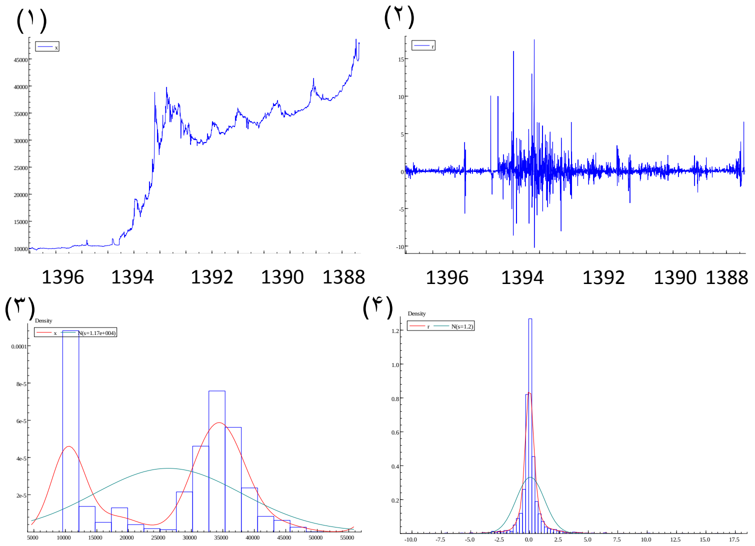

Fig. 1: The return charts and the daily price of the unofficial exchange rate and the histogram related to them

In order to determine the stationary of time series, the root tests of Dickey Fuller, Phillips-Perron and Zivot-Andrews were implemented on the data. According to the results of the tests presented in Table1, the time series of exchange rates is not stationary at the level. However, after a first difference of each of the three criteria, the stationary of the series was confirmed at all levels of significance.

Table 1: Results of the unit root test for the daily price of the unofficial exchange rate

test	At level		At first difference	
	intercept	Trend and intercept	intercept	Trend and intercept
Dickey Fuller	0/7798	-1/0528	-41/1767	-41/1711
Phillips-Perron	-0/8257	-1/2222	-47/9558	-47/9481
Zivot-Andrews	-5/0213	-4/9500	-22/0449	-23/3675

Source: Researcher Computations

After confirming the stationary and implementing the heteroscedasticity test on the data, it is time to estimate the discrete and continuous model parameters. According to the results from the estimation, the GARCH (1,1) model was successful in capturing the volatility clustering behavior, because the coefficients for ARCH and GARCH were statistically significant. Therefore, the GARCH (1,1) discrete-time model is a suitable candidate for conditional variance modeling. While the discrete model parameters were estimated, the continuous model parameters were also estimated based on the relationships

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between the discrete and continuous models using quasi-maximum likelihood method.

Table 2: Results of discrete model parameters estimation

Estimation of GARCH(1,1) model parameters		
β	λ	$\boldsymbol{\delta}$
0.1071	0.4359	0.5559

Table 3: Results of continuous model parameters estimation

Estimation of COGARCH(1,1) model parameters		
$\hat{\beta}$	$\hat{\eta}$	$\hat{\boldsymbol{\varphi}}$
0.1496	0.0629	0.0539

Source: Researcher Computations

After estimating the parameters, using the recursive equations for the volatility process, the recursive estimation of the COGARCH (1,1) volatility process along with the conditional volatility of the GARCH(1,1) is shown in Figure 2. As it can be seen from Figure 2, the COGARCH model (1,1) was successful in capturing volatility and the close relationship between the two models was well demonstrated.

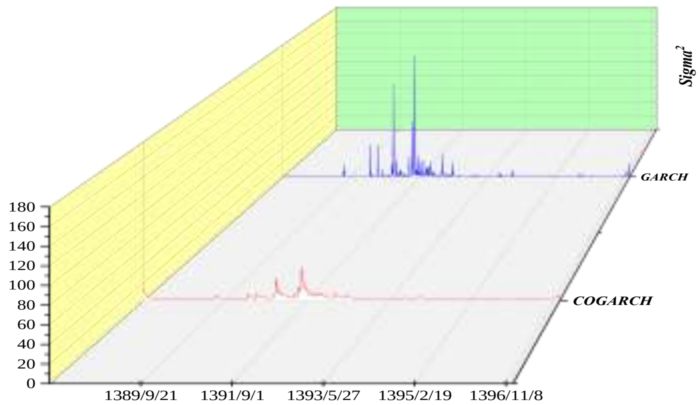

Fig. 2: Conditional volatility of GARCH and COGARCH models

Now we want to examine the efficiency of this continuous model compared to discrete GARCH model. Therefore, we obtained and compared the error of both models using three RMSE, MAE and MHSE error measurement tools. The predictive power of the two models can be compared using these errors. Because of the volatility clustering feature, the MHSE criterion was more suitable for error measurement and it was found that the COGARCH (1,1) model could provide a better prediction of conditional volatility.

Table 4: The results of the volatility estimation error

	RMSE	MAE	MHSE
GARCH(1,1)	1.7187	1.0249	1.8535
COGARCH(1,1)	2.1901	1.8782	1.1442

Source: researcher computation

5. Conclusion

In this paper, we tried to model the volatility in the Iranian exchange market using a stochastic differential equation under the Levy process called the COGARCH model(1,1), and to provide a new application of continuous models. In line with this goal, by estimating the parameters of discrete GARCH model (1,1) and making this discrete model a candidate for conditional variance modeling, we used it to obtain a continuous version and estimate the parameters of the model. The results of comparison between the two models showed that the continuous model also captured the volatility well and the volatilities in the Iranian exchange market follow the model. Therefore, the COGARCH model (1,1) produced very good results in modeling the series of exchange rate returns, since the model obtained the attributes of the volatility process and gave better conditional volatility estimates than those of their discrete-time counterparts. To find out the priority of these two models, we used the error comparison of the two models to compare their efficiency and predictive power. According to the results, the MHSE criterion introduced the COGARCH model(1,1) as more suitable for predicting the conditional volatility and display features of the Iranian exchange market.

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